Synopsis for PhD programme

Title of the research:

Applications of Sinc Wavelet Technique to Some Ecological Models with Allee Effect and Diffusion

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1 Introduction

In long ago scheme of approximating a function in the sinc basis was invented by the outstanding mathematician E.T. Whittaker is known in the mane of Whittaker Cardinal expansion.Recent past the same mathematical tool used by the engineers in their studies of signal processing in the name of Shannon wavelet analysis.Recently the sinc method have been remamped by Frank Stenger and his school widely used in various fields of numerical analysis, solution of integral, ordinary differential and partial differential equations . Sinc-Galerkin is one of the sinc methods that used for solving boundary value problems with singular solutions[1,2,3,4]. Despite most of the numerical methods, sinc-Galerkin method comprehends problems that have singular solutions. Conventional form of these methods is single exponential or double exponential transformation. There are several advantages to using approximations based on sinc numerical methods. First, unlike most numerical techniques, it is now well-established that they are characterized by exponentially decaying errors . Second, they are highly efficient and adaptable in handling problems with singularities . Finally, due to their rapid convergence, sinc numerical methods do not suffer from the common instability problems associated with other numerical methods .We try to solve some nonlinear differential equation describing population models with Allee effect and diffusion using sinc numerical technique[5].

2 Allee Effect

Warder C. Allee brought attention to the possibility of a positive relationship between (components of) individual fitness and population size more than 50 years ago. Since then, it has been widely recognized that individuals of many species benefit from the presence of conspecifics. The phenomenon, broadly referred to as the Allee effect, attracted a lot of attention during the last decade with the rise of conservation biology. Based on a widespread evidence in natural populations, several mechanisms have been hypothesized to invoke the Allee effect[6].

The generally accepted definition of Allee effect is positive density dependence, or the positive correlation between population density and individual fitness. It can sometimes be referred to as "undercrowding".

The main cause of the Allee effect is the difficulty in finding mates between the individuals of a species at low population densities. Other causes may be reduced defense against predators, special trends of social behavior, etc. The dynamical description by differential equations is valid in the case of populations with distinct, non-overlapping generations, for example some insect populations. Such a dynamical description has often been used in population dynamics in connection to logistic type maps.

2.1 Single-species Models of the Allee Effect:

Here, we consider a single population whose growth in a homogeneous environment described either by the ordinary differential equation

$$\frac{dN}{dt} = Ng(N)$$

the function $g(N)$ denotes the density-dependent per capita growth rate. The per capita growth rate describing the Allee effect at low population sizes is hump-shaped.
The reviewed heuristic models are summarized[7].

No l	per capita growth rate	Parameter constrain	Scenario	Reference
1	$g(N) = r - b(N-a)^2$	$a, b > 0$	UE,ES,US	Edelstein-Keshet
2	$g(N) = r(1-\frac{N}{K})(\frac{N}{K} - \frac{A}{K})$	$r, K > 0$	$ES[A > 0], US[A < 0]$	Lewis & Kareiva
3	$g(N) = r(1-\frac{N}{K})(\frac{N}{K} - 1)$	$r, K, A > 0$	ES	Wilson & Bosser
4	$g(N) = r(1-\frac{N}{K})r(1-\frac{A+C}{N+C})$	$r, K, C > 0$	$ES[A > 0], US[A < 0]$	Gruntfest et al
5	$g(N) = b + \frac{a-N}{1+cN}N$	$a, c > 0$	$UE, ES[b < 0]$	Takeuchi

2.2 Two Species Competition Model Involving Allee Effect

The two species population model is given by

$$\frac{dN_1}{dt} = f_1(N_1, N_2) \tag{1}$$

$$\frac{dN_2}{dt} = f_2(N_1, N_2). \tag{2}$$

Here $f_1(N_1, N_2)$ and $f_2(N_1, N_2)$ are function of N_1 and N_2 that describe growth rate of species including Allee effect and interaction between them.

2.3 Reaction-Diffusive Model

Spatio-temporal models can take Allee effect into account as well

$$\frac{\partial N_1}{\partial t} = D_1\frac{\partial^2 N_1}{\partial x^2} + f_1(N_1, N_2) \tag{3}$$

$$\frac{\partial N_2}{\partial t} = D_2\frac{\partial^2 N_2}{\partial x^2} + f_2(N_1, N_2). \tag{4}$$

Here D_1 and D_2 are diffusion coefficient.

3 Approximation Theory and Sinc Numeric Technique

One of the basic question of approximation theory is the following-Given a function f, a norm and a class S of approximating function functions, how well can f be approximated in this norm

by function in S? This question becomes especially interesting for non-smooth functions[8-11].

First denote the set of all integers, The set of real numbers, and the set of complex numbers by Z, R and C, respectively.

$\bullet sinc(z) = sin(\pi z/\pi z), z \in C$

Note that $|sinc(x)| \leq 1$ for any $x \in R$

$\bullet S(k,h)(z) = sinc[(z-kh)/h], z \in c, h > 0$

$\bullet C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(hk)S(k,h)(x), h > 0$

Here, $C(f,h)(x)$ is called the Whittakercardinal expansion of $f(x)$ whenever the series converges.

$\bullet C_N(f,h)(x) = \sum_{k=-N}^{N} f(kh)S(k,h)(x)$

Approximation can be constructed for infinite, semi-infinite, and finite intervals.To construct approximation on the interval (a,b), which are used in this paper,consider the transformation

$\phi = ln(\frac{x-a}{b-x})$.

The basis function on (a,b) are given by

$S(j,h) \circ \phi(x) = sinc(\frac{\phi(x)-jh}{h})$.

Notice that these functions exhibit Kroneccker delta behavior on the grid points $x_k \in (a,b)$ defined by

$x_k = \phi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}}$.

The "mesh size" h represents the mesh size in D_d for the uniform grids (hk), $k = 0, \pm 1, \pm 2, \pm 3$ .....The sink grid points $z_k \in (a,b)$ in D_E will be denoted by x_k because they are real. The sinc-Galerkin method requires that the derivatives of composite sinc functions be evaluated at the nodes[2].

Let ϕ be the conformal one to one mapping of the simply connected domain D_E onto D_d given by (1). Then

$$\delta_{jk}^{(0)} = [S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} 0 & \text{if } j=k \\ \frac{(-1)^{k-j}}{k-j} & \text{if } j\neq k \end{cases}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3} & \text{if } j=k \\ \frac{-2(-1)^{k-j}}{(k-j)^2} & \text{if } j\neq k \end{cases}.$$

3.1 Nonlinear two-point boundary value problems

Consider a nonlinear, second-order BVP of the form

$$\mu_2(x)y''(x) + \mu_1(x)y'(x) + \mu_0(x)y(x) + P(x)H(y) = f(x), a < x < b$$

where $P(x)$ is an analytic function and $H(x)$ is any analytic function of y which has a power series expansion. Consider $H(y) = y^n$, where n is an integer, or a function[2].

3.1.1 The sinc-Galerkin method

The approximate solution for $y(x)$ is represented by the formula

$$y_m(x) = \sum_{j=-N}^{N} c_j S_j(x), m = 2N+1.$$

The unknown coefficient c_j in the equation are determined by orthogonalizing the residual against each basis function w.r.t. to the inner product[4]

$$< \mu_2(x)y''(x), S_k > + < \mu_1 y'(x), S_k > + < \mu_0(x)y(x), S_k > + < P(x)y''(x), S_k >=< f(x), S_k >.$$

3.1.2 The sinc-collocation method

Consider solution of BVP be

$$y_m(x) = \sum_{j=-N}^{N} c_j S(j,h)\circ\phi(x), m = 2N+1.$$

Then the discrete sinc-collocation system for the determination of unknown coefficient is given by

$$\sum_{j=-N}^{N}[\sum_{i=0}^{2} g_i(x_k)\frac{(-1)^i\delta_{kj}^{(i)}}{h^i}]c_j + P(x_k)c_k^n = f_k, k = -N, -N+1, ...N.$$

The system takes the matrix form

$$\Lambda c + \phi c^n = \Theta. \tag{5}$$

In both cases, the differential equation reduces to algebraic equation.

4 Summary and Conclusions

Our objective here is to apply the sinc-Galarkin and sin collocation method to get approximate solution of nonlinear differential equations describing population model with Allee effect and diffusion. Also, we can improve the accuracy of the solution by selecting the appropriate shape parameters and selecting the appropriate values of N.

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